METRIC AND TOPOLOGICAL SPACES: RE-EXAM 2022/23

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Problem 1 (20%). The discrete metric d_0 on $\mathbb{R}$ attains exactly two distinct values. Can a metric ϱ on $\mathbb{R}$ attain exactly three distinct values? (If not, prove; if yes, give example)

Problem 2 (20%). Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space and $\varnothing \neq A \subseteq \mathcal{X}$ its subset. Prove that the interior $\operatorname{Int}(A) = \{a \in A \mid \exists\, \varepsilon(a) > 0, \mathsf{B}_{\varepsilon}^{d_{\mathcal{X}}}(a) \subseteq A\}$ is open in $\mathcal{X}$.

Problem 3 (20%). Let $\mathcal{X}$ be a space and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mathcal{Y}$ is connected and $\mathcal{Y} \subseteq \mathcal{Z} \subseteq \overline{\mathcal{Y}}$, then $\mathcal{Z}$ is connected. (prove)

Problem 4 (20%). Prove that the diameter of every compact metric space $(\mathcal{X}, d_{\mathcal{X}})$ is finite.

(By definition, $\operatorname{diam}(\varnothing) = 0$ and $\operatorname{diam}(S) = \sup_{x,y \in S} d_{\mathcal{X}}(x, y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Problem 5 (20%). Give an example of complete metric space $(\mathcal{X}, d_{\mathcal{X}})$ and map $f\colon \mathcal{X} \to \mathcal{X}$ such that for all $x, y \in \mathcal{X}$ we have $d_{\mathcal{X}}(f(x), f(y)) < d_{\mathcal{X}}(x, y)$ but f is *not* a contraction.

(if $x \neq y$)